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THE INFLUENCE OF SMALL SURFACE IRREGULARITIES ON THE STRESS STATE OF A BODY AND THE ENERGY BALANCE FOR A GROWING CRACK†

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Using an averaging procedure two terms of the asymptotic form for the stress and strain state of a plane body with rapidly oscillating boundary $\Gamma_0(h)$ that imitates a rough surface are constructed. In the investigation of the boundary layer new boundary conditions arise on the limiting smooth contour $\Gamma_0=\Gamma_0(0)$. Two formulations of the problem are proposed, which take into account a correction term for the asymptotic form away from Γ_0 and yield an approximation with increased accuracy $O(h^2)$ e.g. for the potential energy of the strains. The problem concerning the strains of a domain with a very winding crack is considered (under the assumption that the edges have no contact points). Near the tips of the crack there is an additional angular boundary layer, which, in particular, makes it necessary to specify longitudinal concentrated compressive forces at the tips of the limiting crack, which can be represented by a mathematical cut (the error in such a model of a winding crack amounts to $O(h^2)$ only). The balance of energy (in the framework of the Griffith hypothesis) for a developing crack provides a criterion for fracture involving the magnitude of the load (compression) along the crack.

1. THE ASYMPTOTIC FORM OF THE STRESS AND STRAIN STATE IN THE VICINITY OF A ROUGH SURFACE

LET Ω be a domain in $\mathbf{R}^2$ with boundary $\partial\Omega$ consisting of contours $\Gamma_0,\ldots,\Gamma_J$. Also let Γ_0 be a smooth (of class C^∞) simple closed arc of length l_0, and let (n,s) be the natural local coordinates connected with that arc. We will denote by N a large natural number, and we will set $\Gamma_0(h)=\{(n,s)\colon\ s\in[0,l_0],\ \ n=h\gamma(s,h^{-1}s)\}$, where $h=l_0N^{-1}$ is a small parameter and $\gamma\in C^\infty([0,l_0]\times[0,1])$ is a function periodic both in s and $t=h^{-1}s$ with periods l_0 and 1, respectively. The domain $\Omega(h)$ bounded by $\Gamma_0(h)$ and $\Gamma_1,\ldots,\Gamma_J$ has a rapidly oscillating rough boundary. We can reduce the characteristic dimension of Ω to unity by scaling. Then the Cartesian coordinates $\mathbf{x}=(x_1,x_2)$, the coordinates (n,s), as well as l_0 and h become dimensionless.

We will consider the following plane problem of the theory of elasticity:

$$\mathbf{L}(\nabla_x)\,\mathbf{u}^h(\mathbf{x})\equiv\mu\nabla_x\cdot\nabla_x\mathbf{u}^h(\mathbf{x})+(\lambda+\mu)\,\nabla_x\nabla_x\cdot\mathbf{u}^h(\mathbf{x})=0,\quad \mathbf{x}\in\Omega(h) \tag{1.1}$$

$$\sigma^{(n)}(\mathbf{u}^h;\mathbf{x})=-\mathbf{p}^i(\mathbf{x}),\quad \mathbf{x}\in\Gamma_i,\quad i=1,\ldots,J \tag{1.2}$$

$$\sigma^{(n_h)}(\mathbf{u}^h;\ \mathbf{x})=-\mathbf{p}^0(s,h^{-1}s),\quad \mathbf{x}\in\Gamma_0(h) \tag{1.3}$$

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Here $\mathbf{u}^h$ is the displacement vector, $\nabla_x = \text{grad}$, λ and μ are the Lamé coefficients, $\boldsymbol{\sigma}(\mathbf{u}^h)$ is the stress tensor, $\boldsymbol{\sigma}^{(n)} = \boldsymbol{\sigma}\mathbf{n}$, $\mathbf{n}$ is the inward normal (with respect to Ω) unit vector to $\partial\Omega\backslash\Gamma_0$, $\mathbf{n}_h$ is the normal unit vector to $\Gamma_0(h)$, $\mathbf{p}^0$, and $\mathbf{p}^1, \ldots, \mathbf{p}^J$ are the external loads (their principal vector and momentum are equal to zero) and $\mathbf{p}^0 \in C^\infty([0, l_0] \times [0, 1])$ is a periodic function.

We shall find the asymptotic form of $\mathbf{u}^h$ as $h \to 0$ using the well-known method of averaging (see [1–3] and other papers). Away from $\Gamma_0(h)$ the solution can be represented as a series $\mathbf{v}^0(\mathbf{x}) + h\mathbf{v}^1(\mathbf{x}) + \ldots$. In the vicinity of $\Gamma_0(h)$ there occurs the effect of an exponential boundary layer determined by the solution of the problem in the domain $\Pi(s) = \{\eta \in \mathbf{R}^2 : \eta_1 \in (0, 1), \eta_2 > \gamma(s, \eta_1)\}$, which depends on the parameter $s \in [0, l_0]$. The boundary layer can be sought in the form $h\mathbf{w}^1(s, \eta) + h^2\mathbf{w}^2(s, \eta) + \ldots$, where $\eta_1 = h^{-1}s$, $\eta = h^{-1}n$. The function $\mathbf{v}^k$ satisfies the equations

$$\mathbf{L}(\nabla_x)\,\mathbf{v}^{(k)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega$$
$$\sigma^{(n)}(\mathbf{v}^{(k)}; \mathbf{x}) = -\delta_{k,0}\mathbf{p}^i(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i,\ i = 1, \ldots, J \tag{1.4}$$

The boundary conditions for $\mathbf{v}_k$ on Γ_0 can be obtained by studying the problem concerned with the boundary layer. We substitute the sum $\mathbf{v}^0 + h\mathbf{w}^1$ into (1.1) and (1.3). Collecting the factors of h^{-1} in (1.1) and h^0 in (1.3), we obtain the following relations, which we supplement by adding the periodicity conditions for $\mathbf{w}^1$ with respect to $\eta_1 \in [0, 1]$:

$$\mathbf{L}(\nabla_\eta)\,\mathbf{w}^1(s, \eta) = 0, \quad \eta \in \Pi(s) \tag{1.5}$$

$$\tau^{(\nu)}(w^1; s, \eta) = -\mathbf{p}^0(s, \eta) - \boldsymbol{\sigma}(v^0, s, 0)\,\boldsymbol{\nu}(s, \eta), \quad \eta \in \pi(s) \tag{1.6}$$

$$(\partial^k \mathbf{w}^1/\partial\eta_1{}^k)(s, 0, \eta_2) = (\partial^k \mathbf{w}^1/\partial\eta_1{}^k)(s, 1, \eta_2), \eta_2 \geqslant \gamma(s, 0), \quad k = 0,1 \tag{1.7}$$

Here ν is the inward normal unit vector to the face $\pi(s) = \{\eta : \eta_1 \in [0, 1], \eta_2 = \gamma(s, \eta_1)\}$ of the "half-strip" $\Pi(s)$, $\boldsymbol{\tau}(\mathbf{w}^1)$ is the stress tensor computed by regarding η as Cartesian coordinates, and $\boldsymbol{\sigma}(\mathbf{v}^0; s, 0)$ are the values of $\boldsymbol{\tau}(\mathbf{v}^0 : x)$ on Γ_0. The conditions for the existence of a solution of the problem that vanishes exponentially at infinity (the vanishing principal load vector) have the form

$$\sigma^{(n)}(\mathbf{v}^0; s) = -\mathbf{P}^0(s), \quad \mathbf{x} \in \Gamma_0 \tag{1.8}$$

Here $\mathbf{P}^0(s)$ is the principal vector $\mathbf{P}^0(s, \cdot)$ on the arc $\pi(s)$.

The second term $\mathbf{v}^{(1)}$ of the outer expansion satisfies (1.4), and the boundary conditions on Γ_0 can be obtained by considering the term $\mathbf{w}^2$ of the boundary layer type. To simplify the formulae, we assume that the contour $\Gamma_0(h)$ is free of stress [i.e. $\mathbf{p}^0 = 0$ in (1.3)]. Then, taking into account the homogeneous conditions (1.3), we find that (1.6) takes the form $\boldsymbol{\tau}^{(\nu)}(\mathbf{w}^1) = -\sigma_{ss}(\mathbf{v}^0)(\nu_1, 0)$ on $\pi(s)$. We denote by $\mathbf{W}(s, \eta)$ the solution of (1.5) that satisfies conditions (1.7) and the equality

$$\boldsymbol{\tau}^{(\nu)}(\mathbf{W}; s, \eta) = -(\mu\nu_1(s, \eta), 0), \quad \eta \in \pi(s) \tag{1.9}$$

It is obvious that $\mathbf{w}^1 = \mu^{-1}\mathbf{W}\sigma_{ss}(v^0)$. We introduce the non-negative number

$$\mathbf{E}(\mathbf{W}; \Pi(s)) = -\int_{\pi(s)} \boldsymbol{\tau}^{(\nu)}(\mathbf{W}) \cdot \mathbf{W}\, dl_\eta \equiv \mu^2 A^{-1} b(s)$$

$$b(s) = \mu^{-2}(2\mu + \lambda)^{-1} \sum_{j,k=1}^{2} \int_{\Pi(s)} \{2(\mu + \lambda)\,\tau_{jk}(\mathbf{W})^2 - \lambda\tau_{jj}(\mathbf{W})\,\tau_{kk}(\mathbf{W})\}\, d\eta$$

$$A = 4\mu(\mu + \lambda)(2u + \lambda)^{-1} \tag{1.10}$$

We also introduce the matrix-valued differential operators $\mathbf{L}^1$, $\mathbf{T}$, and $\mathbf{B}$ defined by the equalities ($k(s)$ is the curvature of the contour at s)

$$\mathbf{L}^1 = \begin{Vmatrix} 2(\lambda+2\mu)\mathrm{D}\partial_1 - \mu k\partial_2 & (\lambda+\mu)(\mathrm{D}\partial_2 - k\partial_1) \\ (\lambda+\mu)(\mathrm{D}\partial_2 + k\partial_1) & 2\mu\mathrm{D}\partial_1 - (\lambda+2\mu)k\partial_2 \end{Vmatrix}$$

$$\mathbf{B} = \begin{Vmatrix} \mu\gamma\partial_n{}^2 - (2\mu+\lambda)\gamma_s\partial_s & \mu\gamma\partial_s\partial_n - \gamma_s(\lambda\partial_n - (2\mu+\lambda)k) \\ \lambda\gamma\partial_s\partial_n - \mu\gamma_s(\partial_n + k) & (2\mu+\lambda)\gamma\partial_n(\partial_n - k) - \lambda\gamma k\partial_n - \mu\gamma_s\partial_s \end{Vmatrix}$$

$$\mathbf{T} = \begin{Vmatrix} \mu k(1-\gamma\partial_2) - (2\mu+\lambda)(\gamma_1 D - \gamma_s\partial_1) & (2\mu+\lambda)\gamma_1 k + \mu\partial_s - \lambda\gamma_1\partial_2 \\ \lambda\partial_s - \mu(\gamma_1 k + \gamma_s\partial_2) & -\lambda k - (2\mu+\lambda)k\gamma\partial_2 - \mu(\gamma_1 D + \gamma_s\partial_s) \end{Vmatrix}$$

$$\partial_n = \frac{\partial}{\partial n},\ \partial_s = \frac{\partial}{\partial s},\quad \mathrm{D} = \partial_s + \eta_2 k\partial_1,\quad \partial_j = \frac{\partial}{\partial \eta_j},\quad \gamma_s = \partial_s\gamma,\ \gamma_1 = \partial_1\gamma$$

We substitute the sum $\mathbf{v}^0 + h(\mathbf{v}^1 + \mathbf{w}^1) + h^2\mathbf{w}^2$ into (1.1) and (1.3), and we collect the factors multiplying h^0 and h^1, respectively. Taking (1.4)–(1.8) into account, we find that $\mathbf{w}^2$ can be obtained from (1.7) and the equations

$$\mathbf{L}\,(\nabla_\eta)\,\mathbf{w}^2\,(s,\eta) = -\mathbf{L}^1\,(\eta, s, \nabla_\eta, \partial_s)\,\mathbf{w}^1\,(s,\eta),\ \eta \in \Pi\,(s) \tag{1.11}$$

$$\boldsymbol{\tau}^{(\nu)}\,(\mathbf{w}^2; s, \eta) = -\mathbf{T}\,(\eta, s, \nabla_\eta, \partial_s)\,\mathbf{w}^1\,(s,\eta) - \boldsymbol{\sigma}\,(\mathbf{v}^1; s, 0)\,\boldsymbol{\nu}\,(s,\eta) + B\,(\eta, s, \partial_s, \partial_\eta)\,\mathbf{v}^0\,(s, 0) \tag{1.12}$$

Using the Betti formula, it can be established that problem (1.11), (1.12), (1.7) is solvable in the class of functions that vanish at infinity in the case when $\mathbf{v}^{(1)}$ satisfies the boundary conditions

$$\boldsymbol{\sigma}^{(n)}\,(\mathbf{v}^{(1)}; x) = \partial_s\,\{(m\,(s) + b\,(s))\,\sigma_{ss}\,(v^0; x)\,\mathbf{s}\},\ \mathbf{x} \in \Gamma_0 \tag{1.13}$$

$$m\,(s) = \int_0^1 \gamma\,(s, \eta_1)\,d\eta_1 \tag{1.14}$$

Here $\mathbf{s}$ is the unit tangent vector corresponding to the positive orientation of Γ_0, and b is defined by (1.10).

To determine $b(s)$, we have to solve problems (1.5), (1.7), (1.9) concerned with the deformation of the domain $\Pi(s)$. If $\gamma(s,\eta) = \delta\gamma_0(s,\eta)$, where $0<\delta$ is a small parameter, i.e. $\Pi(s)$ differs little from the half-strip, then one can derive the asymptotic formula

$$b\,(s) = \delta\lambda \sum_{j=1}^{\infty} [2\pi j\,(2\mu+\lambda)]^{-1}\,(c_{1j}{}^2 + c_{2j}{}^2) + O\,(\delta^2)$$

in which c_{1j} and c_{2j} are the coefficients of the Fourier expansion of the function $\eta \to (\partial\gamma\ /\partial\eta_1)\,(s,\eta_1)$ in $\cos j\eta_1$ and $\sin j\eta_1$.

2. THE "EQUIVALENT" PROBLEM IN A DOMAIN WITH A REGULARLY PERTURBED BOUNDARY

We will assume that γ is independent of the fast variable $h^{-1}s$, i.e. $\gamma(s, h^{-1}s) = \gamma_*(s)$, and we will consider a domain with regularly perturbed boundary (without oscillations). All the computations of the previous section remain valid, but become simpler, since there are no terms of the boundary-layer type. In particular, (1.10) and (1.14) are equal to 0 and $\gamma_*(s)$, respectively.

Therefore, the sum $m+b$ in the boundary condition (1.13) can be replaced by γ_*. Thus, as in [4], it is possible to find a problem in a domain with regularly perturbed boundary that is asymptotically equivalent to problem (1.1)–(1.3) [with accuracy $O(H^2)$]. In other words, one can take the roughness of the surface into account by constructing a smooth surface that is equivalent in a certain sense.

We denote by Ω_* the domain bounded by the contours $\Gamma_{0h} = \{x : s\in[0, l_0), n = h(m(s)+b(s))\}$ and $\Gamma_1, \ldots, \Gamma_J$, and we consider the problem concerning the deformation of Ω_* under the action of the loads $\mathbf{p}^0 = 0$, $\mathbf{p}^1, \ldots, \mathbf{p}^J$ (it is assumed that there are no loads on $\Gamma_0(h)$ or Γ_{0h}). According to what has been said above, the solution $\mathbf{u}^*$ of the problem in question is identical with the solution $\mathbf{u}^h$ of (1.1)–(1.3) with accuracy $O(h^2)$ outside a neighbourhood of Γ_0. Hence, in particular, we obtain the relation

$$\mathbf{U}(\mathbf{u}^h; \Omega(h)) = {}^1/_2\mathbf{E}(\mathbf{u}^h; \Omega(h)) - \int_{\partial\Omega(h)} \mathbf{p}\cdot\mathbf{u}^h\,ds = -{}^1/_2 \int_{\partial\Omega(h)} \mathbf{p}\cdot\mathbf{u}^h\,ds =$$
$$= -{}^1/_2 \int_{\partial\Omega_*} \mathbf{p}\cdot\mathbf{u}^*\,ds + O(h^2) = \mathbf{U}(\mathbf{u}^*; \Omega_*) + O(h^2) \tag{2.1}$$

for the strain energy.

In turn, the latter relation means that the frequencies $\omega_1(h)\leqslant\omega_2(h)\leqslant\ldots$ and $\omega_1^*\leqslant\omega_2^*\leqslant\ldots$ of the characteristic oscillations of $\Omega(h)$ and Ω_* satisfy the inequalities $|\omega_j(h)-\omega_j^*|\leqslant c_j h^2$, where $j = 1, 2, \ldots$.

Another way of constructing the problem involving two terms of the asymptotic expansion consists in assigning some additional strain energy to Γ_0. We shall show how this can be done. Since $\boldsymbol{\sigma}^{(n)}(\mathbf{v}^0) = 0$ on Γ_0, the equality (1.13) can be transformed into

$$\boldsymbol{\sigma}^{(n)}(\mathbf{v}^{(1)}; \mathbf{x}) = A\partial_s\{(m(s)+b(s))(\partial_s v_s^0(\mathbf{x}) - k(s)v_n^0(\mathbf{x}))\mathbf{s}\}, \quad \mathbf{x}\in\Gamma_0.$$

The field $\mathbf{v}^*$, which approximates $\mathbf{v}^{(0)}+h\mathbf{v}^{(1)}$ with an error $O(h^2)$, can now be defined as the solution of (1.4) with $k = 0$ that satisfies the boundary condition

$$\boldsymbol{\sigma}^{(n)}(\mathbf{v}^*; \mathbf{x}) - Ah\partial_s\{(m(s)+b(s))(\partial_s v_s^*(x) - k(s)v_n^*(x))\mathbf{s}\} =$$
$$= 0, \quad \mathbf{x}\in\Gamma_0 \tag{2.2}$$

Integrating by parts in Ω_0 and on Γ_0, we conclude have problem (1.4), (2.2) is equivalent to finding the minimum of the functional

$${}^1/_2\mathbf{E}(\mathbf{v}^*; \Omega_0) + {}^1/_2 Ah\int_{\Gamma_0}(m+b)\,|\partial_s v_s^* - kv_n^*|^2\,ds - \int_{\partial\Omega_0\setminus\Gamma_0}\mathbf{p}\cdot\mathbf{v}^*\,ds \tag{2.3}$$

in the subspace $\{\mathbf{v}^*\in W_2^1(\Omega_0): \partial s v_s^* - kv_n^*\in L_2(\Gamma_0)\}$.

For the problem of minimizing the functional to be well posed, one has to require that the second term in (2.3) be non-negative. This gives rise to the constraint $m+b\geqslant 0$ on Γ_0, which restricts the field of applications of the second formulation of the joint problem. It is, of course, possible to combine both methods, one involving regular perturbations of the boundary, and the other one based on specifying additional strain energy concentrated on Γ_0.

3. A DOMAIN WITH A CRACK WITH ROUGH EDGES (WITHOUT CONTACT POINTS)

In the previous section Γ_0 was assumed to be a contour with a smooth boundary. We shall now

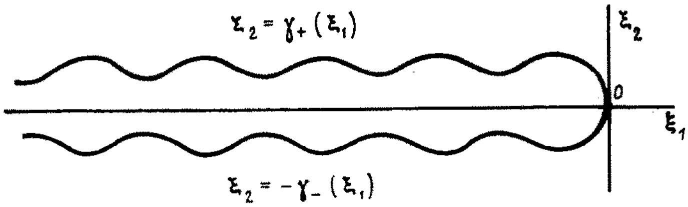

FIG. 1.

assume that Γ_0 is the interval $\{\mathbf{x}: x_2 = 0, |x_1| \leq \frac{1}{2} l_0\}$, whose edges we denote by $\Gamma_0^{\pm}$. Let us describe the contour $\Gamma_0(h)$. Outside a neighbourhood of the end-points $P^{\pm}$ of Γ_0 the contour is given by the arcs $\Gamma_0 \pm (h) = \{\mathbf{x}: x_2 = \pm h\gamma_{\pm}(h^{-1}x_1)\}$, where $\gamma_{\pm} \in C^{\infty}([0, 1])$ are one-periodic functions, $0 \leq \gamma_0(t) \equiv \gamma_+(t) + \gamma_-(t)$ being the (initial) width of the crack. In the vicinity of $P^{\pm}$ the contour $\Gamma_0(h)$ can be obtained by shrinking the boundary $\Xi^{\pm}$ by a factor of h^{-1} (the enlarged end-point zone; see Fig. 1). The domain $\Xi^{\pm}$ can be obtained by removing from $\mathbf{R}^2$ a domain that coincides with the set $\{\xi = h^{-1}(x_1 - \frac{1}{2} l_0, x_2): \xi_1 < 0, \xi_2 \in [-\gamma_-, \gamma_+]\}$ outside a circle of a large radius R_0. Ξ^- can be defined analogously. If the crack developed along a winding path with no material crumbling away, then $\Xi^{\pm}$ is a plane with a cut along a semi-infinite periodic curve.

We denote by $m_{\pm}$ and $b_{\pm}$ the quantities (1.14) and (1.10) corresponding to the upper and lower edges $\Gamma_0^{\pm}(h)$ of the crack. The equalities (1.13) can be rewritten as follows:

$$\sigma_{12}(\mathbf{v}^{(1)}; x_1, \pm 0) = (m_{\pm} + b_{\pm})\, \sigma_{11}(\partial \mathbf{v}^0/\partial x_1; x_1, \pm 0) \qquad (3.1)$$
$$\sigma_{22}(\mathbf{v}^{(1)}; x_1, \pm 0) = 0, \quad \mathbf{x} \in \Gamma_0$$

Since the stress field $\boldsymbol{\sigma}(\mathbf{v}^0)$ has root-type singularities at $P^{\pm}$, the right-hand side of (3.1) increases, in general, as $r_{\pm}^{-3/2}$ as $r_{\pm} \to 0$. Here $(r_{\pm}, \varphi_{\pm})$ are the polar coordinates with centre $P^{\pm}$, $\varphi_{\pm} \in (-\pi, \pi)$. This fact forces us to make the asymptotic form of $\mathbf{u}_h$ more accurate and to construct an additional boundary layer near $P^{\pm}$:

$$\mathbf{z}^{0\pm} + h^{1/2}\mathbf{z}^{1\pm} + h\mathbf{z}^{2\pm} + \ldots \qquad (3.2)$$

We shall restrict ourselves to considering a neighbourhood of P^+ and we shall omit the indices $\pm$. The solution $\mathbf{v}_0$ admits of the expansion

$$\mathbf{v}^0(x) = (c_1 - c_0 x_1,\ c_2 + c_0 x_2) + \sigma^0 A^{-1}(x_1,\ -\alpha x_2) + r^{1/2}(K_1 \boldsymbol{\Phi}^1(\varphi) +$$
$$+ K_2 \boldsymbol{\Phi}^2(\varphi)) + O(r^{3/2}),\ r \to 0 \qquad (3.3)$$

Here the first term is the rigid displacement, σ^0 is a constant, K_j are the stress intensity factors (SIF), and

$$\alpha = \lambda(2\mu + \lambda)^{-1},\quad \varkappa = (3\mu + \lambda)(\mu + \lambda)^{-1},$$
$$(\Phi_r^1, \Phi_\varphi^1)(\varphi) = (4\mu)^{-1}(2\pi)^{-1/2}((2\varkappa - 1)\cos{}^1\!/_2\varphi - \cos{}^3\!/_2\varphi,\quad \sin{}^3\!/_2\varphi - (2\varkappa +$$
$$+ 1)\sin{}^1\!/_2\varphi).\quad (\Phi_r^2, \Phi_\varphi^2)(\varphi) = (4\mu)^{-1}(2\pi)^{-1/2}(3\sin{}^3\!/_2\varphi -$$
$$- (2\varkappa - 1)\sin{}^1\!/_2\varphi,\quad 3\cos{}^3\!/_2\varphi - (2\varkappa + 1)\cos{}^1\!/_2\varphi)$$

Using the method of joint expansions (see, for example, [5]) and comparing (3.2) with (3.3), we find

$$\mathbf{z}^0 - (c_1, c_2), \quad \mathbf{z}^1(\xi) = K_1\mathbf{Z}^1(\xi) + K_2\mathbf{Z}^2(\xi),$$
$$\mathbf{z}^2(\xi) = c_0(-\xi_2, \xi_1) + \sigma_0\mathbf{Z}^0(\xi) \tag{3.4}$$

The vector-valued functions $\mathbf{Z}^k(\xi)$ are solutions of the problems

$$\mathbf{L}(\nabla_\xi)\,\mathbf{Z}^{(k)}(\xi) = 0, \quad \xi \in \Xi, \quad \sigma^{(\nu)}(\mathbf{Z}^{(k)}; \xi) = 0, \quad \xi \in \partial\Xi \tag{3.5}$$

$$\mathbf{Z}^{(q)}(\xi) = \rho^{1/2}\mathbf{\Phi}^{(q)}(\varphi) + o(1), \quad \rho \to \infty, \quad q = 1, 2 \tag{3.6}$$

$$\mathbf{Z}^{(0)}(\xi) = A^{-1}(\xi_1, -\alpha\xi_2) + o(\rho^{1/2}), \quad \rho \to \infty \tag{3.7}$$

We shall make the asymptotic representations (3.6) and (3.7) more accurate. We first expand the vector-valued function $\mathbf{Z}^0$. A modification of the discussion in Sec. 1 yields

$$\mathbf{Z}^{(0)}(\xi) = A^{-1}(\xi_1, -\alpha\xi_2) + B_0\mathbf{T}^{(1)}(\rho, \varphi) + \chi(\xi_1)\sum_{\pm}\mu^{-1}\mathbf{W}^{\pm}(\xi) + o(1) \tag{3.8}$$

$$\mathbf{T}^{(1)}(\xi) = (T_r^{(1)}, T_\varphi^{(1)}) = [8\pi\mu]^{-1}((\varkappa + 1)\ln r(\cos\varphi, -\sin\varphi) +$$
$$+ ((\varkappa - 1)\varphi\sin\varphi, (\varkappa - 1)\varphi\cos\varphi - 2\sin\varphi)) \tag{3.9}$$

Here B_0 is a constant, $\chi \in C^\infty(\mathbf{R})$ is the cut-off function equal to 1 for $\xi_1 < -2R_0$ and 0 for $\xi_1 > -R_0$, and $\mathbf{W}^{\pm}$ are the solutions of problem (1.5), (1.7) and (1.9) in $\Pi^{\pm} = \{\xi: \xi_1 \in (0, 1), \pm\xi_2 > \gamma_{\pm}(\xi_1)\}$.

Let us compute the constant B_0. To do so, we substitute $\mathbf{Z}^{(0)}$ and $\mathbf{e}^1$ into the Betti formula for the domain $Q_{q,t} \cap \Xi$, where $Q_{q,t}$ is the square $\{\xi: |\xi_j| < q + t\}$ and q is a large natural number. As a result, taking (3.5) into account, we find that

$$\int_{\partial Q_{q,t} \cap \Xi} \sigma^{(n)}(\mathbf{Z}^{(0)}; \xi)\cdot\mathbf{e}^1\, ds_\xi = 0 \tag{3.10}$$

In addition, we integrate (3.10) with respect to $t \in (0, 1)$, and we pass to the limit as $q \to +\infty$. Then, using the asymptotic expression (3.8), we neglect all infinitely small terms. We have

$$0 = -\lim_{q\to\infty}\int_0^1 dt \int_{\partial Q_{q,t}\cap\Xi} \mathbf{e}^1\cdot\Big\{\sigma^{(n)}(\rho^1\mathbf{\Phi}^{(0)}; \xi) + B_0\sigma^{(n)}(\mathbf{T}^{(1)}; \xi) + \chi\sum_{\pm}\sigma^{(n)}(\mathbf{W}^{\pm}; \xi)\Big\}\, ds_\xi =$$
$$= \lim_{q\to\infty}\Big\{-\int_0^1 dt \int_{\partial Q_{q,t}} \mathbf{e}^1\cdot\sigma^{(n)}(\rho^1\mathbf{\Phi}^{(0)} + B_0\mathbf{T}^{(1)})\, ds_\xi +$$
$$+ \int_0^1 dt \int_{-\gamma_-(-q-t)}^{\gamma_+(-q-t)} \sigma_{11}(\rho^1\mathbf{\Phi}^{(0)})\, d\xi_2 - \mu^{-1}\sum_{\pm}\pm\int_0^1 dt \int_{\pm\gamma_{\pm}(-q-t)}^{\pm\infty} \sigma_{11}(\mathbf{W}^{\pm})\, d\xi_2\Big\}$$

We can find each of the integrals I_1, I_2, and $I_{\pm}$ in the latter formula:

$$I_1 = \int_0^1 dt \int_{Q_{q,t}} L(\rho^1\mathbf{\Phi}^{(0)} + B_0\mathbf{T}^{(1)})\cdot\mathbf{e}^1\, d\xi = -B_0\int_0^1 dt \int_{Q_{q,t}} \delta(\xi)\, d\xi = -B_0$$

$$I_2 = \int_0^1 \{\gamma_+(\xi_1) + \gamma_-(\xi_1)\}\, d\xi_1 = \int_0^1 \gamma_0(\xi_1)\, d\xi_1 \equiv m_0 = m_+ + m_-$$

$$\sum_{\pm}\pm I_{\pm} = \sum_{\pm}\int_{\Pi_{\pm}} \sigma_{11}(\mathbf{W}^{\pm})\, d\xi = -\mu^{-1}\sum_{\pm}\int_{\pi_+} \{(2\mu + \lambda)W_1^{\pm}\nu_1^{\pm} +$$
$$+ \lambda W_2^{\pm}\nu_2^{\pm}\}\, ds_\xi \equiv b_0 = b_+ + b_-$$

As a result, we arrive at the equality $B_0 = m_0 + b_0$.

The asymptotic form with increased accuracy for the solutions $\mathbf{Z}^{(k)}$, $k = 1, 2$ has the form

$$\mathbf{Z}^{(k)}(\xi) = \rho^{1/2}\mathbf{\Phi}^{(k)}(\varphi) + \rho^{-1/2}(B_{k1}\mathbf{\Psi}^{(1)}(\varphi) + B_{k2}\mathbf{\Psi}^{(2)}(\varphi)) + $$
$$+\delta_{k,2}\left\{\mu^{-1}\chi(\xi_1)\sum_{\pm}(m_\pm \mp b_\pm)\,\mathrm{W}^\pm(\xi)\frac{\partial}{\partial\xi_1}\sigma_{11}(\rho^{1/2}\mathbf{\Phi}^{(2)}(\varphi))\Big|_{\varphi=\pm\pi} + \right.$$
$$\left. + {}^1/_2\rho^{-1/2}[(b_0 + m_+ - m_-)\,\Upsilon^{(1)}(\ln\rho, \varphi) + (m_0 + b_+ - b_-)\,\Upsilon^{(2)}(\varphi)]\right\} + O(\rho^{-2}), \quad \rho\to\infty \tag{3.11}$$

Here B_{kj} are constants depending on Ξ, which form a (2×2)-matrix B, and

$$(\Psi_r^{(1)}, \Psi_\varphi^{(1)})(\varphi) = (8\pi)^{-1/2}(1+\varkappa)^{-1}(3\cos{}^1/_2\varphi - (2\varkappa+1)\cos{}^3/_2\varphi, \;\; (2\varkappa-1)\sin{}^3/_2\varphi -$$
$$-3\sin{}^1/_2\varphi); \;\; (\Psi_r^{(2)}, \Psi_\varphi^{(2)}) = (8\pi)^{-1/2}(1+\varkappa)^{-1}((2\varkappa+1)\sin{}^3/_2\varphi - \sin{}^1/_2\varphi,$$
$$(2\varkappa-1)\cos{}^3/_2\varphi - \cos{}^1/_2\varphi); \;\; (\Upsilon_r^{(1)}, \Upsilon_\varphi^{(1)})(\ln r, \varphi) = [2(2\pi)^{3/2}\mu]^{-1}\{-2\ln r\,\times$$
$$\times(\sin{}^1/_2\varphi - (2\varkappa+1)\sin{}^3/_2\varphi, \cos{}^1/_2\varphi - (2\varkappa-1)\cos{}^3/_2\varphi) + (4(\varkappa\sin{}^3/_2\varphi - \sin{}^1/_2\varphi) -$$
$$-\varphi(\cos{}^1/_2\varphi + (2\varkappa+1)\cos{}^3/_2\varphi), \; 4(\varkappa\cos{}^3/_2\varphi - \cos{}^1/_2\varphi) + \varphi(\sin{}^1/_2\varphi + (2\varkappa -$$
$$-1)\sin{}^3/_2\varphi))\}; \;\; (\Upsilon_r^{(2)}, \Upsilon_\varphi^{(2)})(\varphi) = [4\mu(2\pi)^{1/2}]^{-1}(-\cos{}^1/_2\varphi -$$
$$-(2\varkappa+1)\cos{}^3/_2\varphi, \; \sin{}^1/_2\varphi + (2\varkappa-1)\sin{}^3/_2\varphi)$$

Using the Betti formula in the region $Q_{q,t}\cap\Xi$ applied to the vectors $\mathbf{Z}^{(k)}$ and $\mathbf{Z}^{(j)}$, one can establish that $\mathbf{B}$ is a symmetric matrix.

Let us now return to considering the field $\mathbf{v}^{(1)}$. Taking relations (3.4), (3.8), and (3.11) into account and equating the inner and outer expansions, we can find additional conditions that must be satisfied by the solutions of (1.4) and (3.1):

$$\mathbf{v}^{(1)}(x) = \sum_{j,k=1}^{2} K_j^\pm B_{jk}^\pm r_\pm^{-1/2}\mathbf{\Psi}^{(k)}(\varphi_\pm) + \sigma_0^\pm(m_0 + b_0)\,\mathbf{T}^{(1)}(r_\pm, \varphi_\pm) +$$
$$+ {}^1/_2K_2^\pm r_\pm^{-1/2}[(b_0 + m_+ - m_-)\,\Upsilon^{(1)}(\ln r_\pm, \varphi_\pm) +$$
$$+(m_0 + b_+ - b_-)\,\Upsilon^{(2)}(\varphi_\pm)] + O(1), \quad r_\pm\to 0 \tag{3.12}$$

Here and henceforth we re-establish the indices $\pm$ denoting the tips of the crack Γ_0. Using the general results of [6, 7], one can verify that problem (1.4), (3.1) and (3.12) is solvable [in other words, the load, which also includes the singular terms appearing on the right-hand side of (3.12), is self-balanced]. Note that the principal vector of the smooth load (3.1) is equal to $(b_0 + m_0)(\sigma_0^+ - \sigma_0^-)$ and is compensated by the concentrated forces specified according to (3.12) at the tip of Γ_0.

We shall determine the asymptotic behaviour of the potential energy $\mathrm{U}(\mathbf{u}^h; \Omega(h))$ of deformations. By analogy with Sec. 2, we find that

$$\mathrm{U}(\mathbf{u}^h; \Omega(h)) = \mathrm{U}(\mathbf{v}^0; \Omega) - {}^1/_2h\int_{\partial\Omega\setminus\Gamma_0}\mathbf{p}\cdot\mathbf{v}^1\,ds_x + O(h^2)$$

To evaluate the latter integral, we will apply the Betti formula in Ω with two discs $B_\delta^\pm$ of small radii and centres at $P^\pm$ removed. We have

$$\int_{\partial\Omega\setminus\Gamma_0}\sigma^{(n)}(\mathbf{v}^0)\cdot\mathbf{v}^1\,ds_x = \sum_\pm \pm\int_{\delta-l_0/2}^{-\delta+l_0/2}\sigma_{12}(\mathbf{v}^1; x_1, \pm 0)\,v_1^0(x_1, \pm 0)\,dx_1 -$$
$$-\int_{\partial B_\delta^\pm}(\sigma^{(r)}(\mathbf{v}^{(0)}; x)\cdot\mathbf{v}^{(1)} - \sigma^{(r)}(\mathbf{v}^{(1)}; x)\cdot\mathbf{v}^{(0)})\,ds_x$$

Using (3.1), (3.3) and (3.13), we finally get

$$\mathbf{U}(\mathbf{u}^h; \boldsymbol{\Omega}(h)) = \mathbf{U}(\mathbf{v}^0; \boldsymbol{\Omega}) - {}^1\!/_2 h (m_0 + b_0) \Big\{ \pi \sum_{\pm} \sigma_0{}^{\pm} c_1{}^{\pm} -$$

$$- \lim_{\delta \to 0} \Big[A \int\limits_{\delta - {}^1\!/_2 l_0}^{-\delta + {}^1\!/_2 l_0} \Big(\Big| \frac{\partial v_1{}^0}{\partial x_1}(x_1, +0) \Big|^2 + \Big| \frac{\partial v_1{}^0}{\partial x_1}(x_1, -0) \Big|^2 \Big) dx_1 +$$

$$+ 8\pi^{-1} A^{-1} \ln \delta \sum_{\pm} K_{2,\pm}^2 \Big] \Big\} + h \sum_{\pm} \sum_{p,q=1}^{2} K_{p,\pm} K_{q,\pm} B_{q}$$

Here $K_{p,\pm}$, $\sigma_0{}^{\pm}$ and $c_1{}^{\pm}$ are the quantities for the right and left end-points of the crack appearing in (3.3).

4. ENERGY BALANCE DURING THE DEVELOPMENT OF THE CRACK

Consider an elastic plane with a winding crack $\Gamma_0(h)$ subject to a biaxial load at infinity, that is, consider the problem

$$\mathbf{L}(\nabla_x) \mathbf{u}^h(\mathbf{x}) = 0, \ \mathbf{x} \in \mathbf{R}^2 \setminus \Gamma_0(h); \ \boldsymbol{\sigma}^{(n)}(\mathbf{u}^h; \mathbf{x}) = 0, \ \mathbf{x} \in \Gamma_0(h), \tag{4.1}$$

$$\sigma_{jj}(\mathbf{u}^h; \ \mathbf{x}) = p_j + o(|\mathbf{x}|^{-1}), \quad j = 1, 2, \ \sigma_{12}(\mathbf{u}^h; \mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| \to \infty \tag{4.2}$$

We will assume that the inequalities $p_2 > 0$ and $p_1 < 0$ are satisfied, which means that the crack is stretched in the transverse direction and compressed in the longitudinal direction. For such loads it is natural to assume that the edges of the crack do not touch each other.

The solution $\mathbf{v}^{(0)}$ of the limiting problem is known (see [8]). We will merely recall that

$$\sigma_{11}(\mathbf{v}^{(0)}; x_1, \pm 0) = p_1 - p_2, \quad K_1{}^{\pm} = (\pi l_0)^{1/2} p_2, \quad K_2{}^{\pm} = 0, \quad \sigma_0{}^{\pm} = \\ = p_1 - p_2 \tag{4.3}$$

We turn to the second term $v^{(1)}$ of the outer expansion. By virtue of (3.1) and (4.3), $\mathbf{v}^{(1)}$ satisfies the relations

$$\mathbf{L}(\nabla_x) \mathbf{v}^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbf{R}^2 \setminus \Gamma_0, \quad \sigma_{12}(\mathbf{v}^{(1)}; \mathbf{x}) = \sigma_{22}(\mathbf{v}^{(1)}; \mathbf{x}) = 0, \\ \mathbf{x} \in \Gamma_0; \quad \sigma_{jk}(\mathbf{v}^{(1)}, \mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| \to \infty, \quad j, k = 1, 2 \tag{4.4}$$

Therefore, $\mathbf{v}^{(1)}$ is non-zero only as a result of the asymptotic conditions (3.12). Introducing some simplifying assumptions, we shall consider in more detail problems (3.5)–(3.7) concerned with the boundary layer. First, we shall assume that the initial width of the crack is zero, i.e. $\gamma(t) \equiv \gamma_+(t) = -\gamma_-(t)$. Next, let $\gamma(t) = -\gamma(t + \frac{1}{2})$, i.e. let the crack have a bend symmetric with respect to the Ox_1 axis. We introduce the domain $\Xi(t) = \mathbf{R}^3 \backslash \{\mathbf{x}: x_1 \leq t, \ x_2 = \gamma(x_1)\}$ depending on the parameter t (Fig. 2). In view of the symmetry of the domain, the corresponding coefficient $B_{21}(t)$ in (3.11) satisfies the formula $B_{21}(t) = -B_{21}(t + \frac{1}{2})$. Thus there exists a number $t_0 \in [0, 1)$ such that $B_{21}(t_0) = 0$. Let us fix this location of the tip of the crack. Now, recalling the equalities $m_+ = -m_-$ and $b_+ = b_- = \frac{1}{2} b_0$, we conclude in accordance with (4.3) and (3.12) that formulae (4.4) should be supplemented by the condition

$$\mathbf{v}^{(1)}(\mathbf{x}) = B_{11} (\pi l_0)^{1/2} p_2 (2r_{\pm})^{-1/2} \boldsymbol{\Psi}(\varphi_{\pm}) + b_0 (p_1 - p_2) \mathbf{T}^{(1)}(r_{\pm}, \varphi_{\pm}) + \\ + O(1), \ r_{\pm} \to 0 \tag{4.5}$$

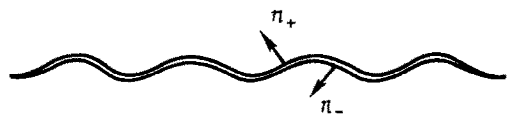

FIG. 2.

Therefore, $\mathbf{v}^{(1)} = (\frac{1}{2}\pi l_0)^{1/2} p_2(\zeta^+ + \zeta^-) + (p_1 - p_2)\zeta_0$, where $\zeta^{\pm}$ is a solution of the homogeneous problem (4.4) that is bounded outside any neighbourhood of the point $(\pm\frac{1}{2}l_0, 0)$ and satisfies the condition $\zeta^{\pm}(x) = r_{\pm}{}^{-1/2}\boldsymbol{\Psi}(\varphi_{\pm}) + o(1)$ as $r_{\pm} \to 0$ (in other words, $\zeta^{\pm}$ are weight functions; see [9, 7]), and where ζ^0 is a solution of the same problem that satisfies the condition $\zeta^0(x) = \mathbf{T}^{(1)}(r_{\pm}, \varphi_{\pm}) + O(1)$ at both tips of the crack Γ_0. The vector-valued functions $\zeta^{\pm}$, ζ^0 are determined by relations of the form [8]

$$\zeta^{\alpha}(x) = \delta_{0,\alpha}[\mathbf{T}^{(1)}(x_1 + {}^1/_2 l_0, x_2) - \mathbf{T}^{(1)}(x_1 - {}^1/_2 l_0, x_2)] + {}^1/_4 \mu^{-1} \langle (\varkappa - 1)\,\mathrm{Re}\, Z_I{}^{0,\alpha}(x) - 2x_2\,\mathrm{Im}\, Z_I{}^{\alpha}(x),\ (\varkappa + 1)\,\mathrm{Im}\, Z_I{}^{0,\alpha}(x) - 2x_2\,\mathrm{Re}\, Z_I{}^{\alpha}(x)\rangle \tag{4.6}$$

$$\alpha = 0, \pm;\ z = x_1 + ix_2$$

$$Z_I{}^{0,\pm}(x) = \pm[(z \pm {}^1/_2 l_0)^{1/2} - (z \mp {}^1/_2 l_0)^{1/2}](z \mp {}^1/_2 l_0)^{-1/2}$$

$$Z_I{}^{\pm}(x) = \mp{}^1/_2 l_0 (z \mp {}^1/_2 l_0)^{-1}(z^2 - {}^1/_4 l_0{}^2)^{-1/2}$$

$$Z_I{}^0 = \partial_z Z_I^{0,0}, \quad Z_I{}^0 = \frac{4}{\pi^2}\frac{\mu l_0}{\lambda + 2\mu}(4z^2 - l_0{}^2)^{-1/2}\int_{-1/2 l_0}^{1/2 l_0}(l_0{}^2 - 4\xi^2)^{-1/2}\frac{d\xi}{z - \xi}$$

As before in Sec. 2, one can state the joint problem, the solution $\mathbf{v}^*$ of which differs from the sum $\mathbf{v}^{(0)} + h\mathbf{v}^{(1)}$ by $O(h^2)$ outside any neighbourhoods of the tips of Γ_0. This formulation consists in finding a displacement field $\mathbf{v}^*$ that satisfies (4.1) and (4.2) in the limiting domain $\mathbf{R}^2\backslash\Gamma_0$ and belongs to the set

$$D = \{\mathbf{v}^*:\ \mathbf{v}^* = \mathbf{v} + hB_{11}(c_+\zeta^+ + c_-\zeta^-) + hb_0c_0\zeta^0,\ c_0,\ c_{\pm} \in \mathbf{R}$$

$$v_1(\mathbf{x}) = v_1(x_1, -x_2),\ v_2(\mathbf{x}) = -v_2(x_1, -x_2),\ c_0,\ c_{\pm} \in \mathbf{R}$$

$$\mathbf{v}(\mathbf{x}) = (a_1^{\pm}, 0) + c_{\pm}r_{\pm}^{1/2}\Phi(\varphi_{\pm}) + A^{-1}c_0(x_1, \alpha x_2) + O(r_{\pm}^{3/2}),\ r_{\pm} \to 0$$

$$\mathbf{v}(\mathbf{x}) = (a_1^{\infty}x_1,\ a_2^{\infty}x_2) + o(|x|^{-1}),\ |x| \to \infty;\ a_j^{\pm},\ a_j^{\infty} \in \mathbf{R}\}$$

A similar problem can be considered as an extension in the weight class of the operator of the original problem of elasticity theory for a plane weakened by a crack Γ_0 (see [10, 11], etc.). This makes it possible to state a correct definition of the energy functional for the joint problem (since the solution is singular, the standard definition is unsuitable) involving an additional energy contribution associated with the tips of the crack. However, here we shall not dwell on unnecessarily rigorous definitions, since the discussion below is intuitively clear and admits of an obvious mechanical interpretation.

We will consider the problem of increasing the length of $\Gamma_0(h)$ by h (the bending period of the crack) at both ends. We will assume that the structure of the end zone remains unchanged. We denote the extended crack by $\Gamma_0'(h)$ and the solutions of the corresponding problems by $\mathbf{u}^{h\prime}$, $\mathbf{v}'$, etc. We assume $|p_1 - p_2|/|p_2|$ to be a large parameter, so that the effect of the stretching the transverse field is of the same order as that of the compressing longitudinal field; in other words $|p - p_2|/|p_2| = O(h^{-1/2})$. We adopt the Griffith hypothesis on the balance of the increments of the surface energy and potential energy. Since the functionals $\mathbf{U}(\mathbf{u}^h;\ \mathbf{R}^2\backslash\Gamma_0(h))$ and $\mathbf{U}(\mathbf{u}^{h\prime};\ \mathbf{R}^2\backslash\Gamma_0'(h))$ are infinite, we consider discs D_R of large radii R and we set

$$\Delta U = \lim_{R\to\infty} \{U(u^h;\ D_R \setminus \Gamma_0'(h)) - U(u^h;\ D_R \setminus \Gamma_0(h))\}$$

as in the Griffith problem [12].

By virtue of (4.2), the Betti formula, and what has been said above about the approximation of the solution $\mathbf{u}^h$ of problem (4.1), (4.2) by the sum $\mathbf{v}^{(0)}+h\mathbf{v}^{(1)}$ (or by the solution $\mathbf{v}^*$ of the joint problem), we have

$$\Delta U = -{}^1/_2 \lim_{R\to\infty} \int_{|x|=R} (p_1 n_1, p_2 n_2)\cdot(\mathbf{u}^h - \mathbf{u}^{h'})\, ds_x =$$

$$= -{}^1/_2 \lim_{R\to\infty} \int_{|x|=R} (p_1 n_1, p_2 n_2)\cdot(\mathbf{v}^* - \mathbf{v}^{*'})\, ds_x + O(h^2)$$

Substituting the solutions $\mathbf{v}^{(j)}$ and $\mathbf{v}^{(j)'}$ already computed [formulae (4.3) and (4.6)] into the above formula, we find that

$$\Delta U = -(8\mu)^{-1} h (1+\varkappa)(\pi l_0 p_2^2 + b_0 (p_1 - p_1)^2) + O(h^2)$$

According to Griffith, the increment $\Delta\Pi$ of the surface energy is proportional to h and equal to $4h\eta$, which means that the energy criterion for fracture $\Delta U+\Delta\Pi\leqslant 0$ (the criterion for triggering the expansion of the crack) is equivalent to the inequality

$$-(8\mu)^{-1}(1+\varkappa)(\pi l_0 p_2^2 + b_0 (p_1 - p_2)^2) + 4\eta \leqslant 0 \tag{4.7}$$

We turn our attention to the presence of the longitudinal component of the external load on the left-hand side of (4.7). If p_1-p_2 is not large and $|p_1-p_2|\ll h^{-1/2}|p_2|$, then the term just mentioned can be neglected. Then (4.7) takes the form of the Griffith energy criterion. The same effect involving the presence of the term p_1-p_2 with a small multiplier is shared by the Novozhilov criterion [13], which consists in verifying the inequality

$$d^{-1} \max_{|\varphi|<\pi} \int_{\Upsilon(d,\varphi)} \sigma_{\varphi\varphi}\, ds \leqslant \sigma_c \tag{4.8}$$

Here d is the structure of the medium, $\Upsilon(d, \varphi_0)$ denotes the interval $\{x: r\in(0,d),\ \varphi=\varphi_0\}$, and σ_c is the theoretical strength limit.

It is known (see e.g. [14]) that the substitution of the two terms [of order $O(r^{-1/2})$ and $O(1)$] of the asymptotic expression for the stress into (4.8) imposes an upper limit on the sum $p_2+cd^{1/2}|p_1-p_2|$. Thus relations (4.8) and (4.7) are asymptotically equivalent if d and h are small quantities of the same order. What has been said above confirms the long-standing opinion that the parameter d is connected with the grain size ρ of the medium: if crack develops avoiding the grains, then the period h of γ is also proportional to ρ.

In the description of the shape of the crack it was assumed that the edges $\Gamma_0^{\pm}(h)$ are given by the equations $x_2=\pm h\gamma_{\pm}(h^{-1}x_1)$ with smooth functions. Of course, smoothness is not essential and the structure of the boundary can be more complex. For example, everything that has been said remains valid for a devloping crack that periodically produces little offshoots.

The formulation of the problems on a winding crack is taken from the paper [15] presented at the 6th All-Union Conference on "Mixed Problems in the Mechanics of Deformable Bodies".

REFERENCES

1. BENSOUSSAN A., LIONS J.-L. and PAPANICOLAOU G., *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam, 1978.
2. ZHIKOV V. V., KOZLOV S. .M, OLEINIK O. A. and KHA T'EN NGOAN, Averaging and G-convergence of differential operators. *Usp. Mat. Nauk* **34,** 5, 65–133, 1979.
3. BAKHVALOV N. S. and PANASENKO G. P., *Averaging of Processes in Periodic Media*. Nauka, Moscow, 1984.
4. ZORIN I. S. and NAZAROV S. A., A boundary effect associated with bending a thin three-dimensional plate. *Prikl. Mat. Mekh.* **53,** 4, 642–650, 1989.
5. IL'IN A. M., *Matching the Asymptotic Expansions for the Solutions of Boundary Value Problems*. Nauka, Moscow, 1989.
6. KONDRAT'YEV V. A., Boundary value problems for elliptic equations in domains with conical or angular points. *Mosk. Mat. Obshch.* **16,** 209–292, 1967.
7. MAZ'YA V. G. and PLAMENEVSKII B. A., On the coefficients in the asymptotic form of the solutions of elliptic boundary value problems in domains with conical points. *Math. Nachr.* **76,** 29–60, 1977.
8. SEDOV L. I., *Mechanics of Continuous Media*, Vol. 2. Nauka, Moscow, 1976.
9. BUECKNER H. F., A novel principle for computing stress intensity factors. *ZAMM* **50,** 9, 528–546, 1970.
10. GORBACHUK V. I. and GORBACHUK M. L., *Boundary Value Problems for Differential-Operator Equations*. Nauk. Dumka, Kiev, 1984.
11. NAZAROV S. A., Self-adjoint extensions of the operator for the Dirichlet problem in weighted functional spaces. *Mat. Sb.* **137,** 2, 224–241, 1988.
12. GRIFFITH A. A., The phenomena of rupture and flow in solids. *Phil. Trans. Roy. Soc. London* **A221,** 2, 163–198, 1921.
13. NOVOZHILOV V. V., On the foundations of the theory of equilibrium cracks in elastic bodies. *Prikl. Mat. Mekh.* **33,** 5, 797–812, 1969.
14. MOVCHAN A. B., MOROZOV N. F. and NAZAROV S. A., Fracture near peak-shaped inclusions during shrink fit. In *Plasticity and Fracture of Rigid Bodies*, pp. 137–145. Nauka, Moscow, 1988.
15. LAVROV N. A. and SLEPYAN L. I., On the problem of the propagation of cracks under compression, Mixed Problems in the Mechanics of Deformable Bodies, Proc. of the 4th All-Union Conference, Part 1, p. 194. Odessa Univ., Odessa, 1989.

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